APUNTES, MANUALES, PRESENTACIONES

Universidad de Jaén
Facultad de Ciencias Sociales y Jurídicas

Mathematics I - Double degree in law and business administration and management

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ASIGNATURA: Matemáticas I

Mathematics I

Double degree in Law and Business Administration and Management

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Adapted from Julio Guerrero's Notes
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Chapter 1: Functions and Continuity

Chapter 1: Functions and Continuity

Introduction

- The concept of function appears in all areas of pure and applied mathematics, physics, chemistry, etc.
- Also in economic analysis there are plenty of functions like demand, supply, cost, production, consumption, etc.
- In this chapter we shall study functions of one variable, illustrating the discussion with some important economic examples.

Function of one variable

A variable is a quantity that may take different values.

One variable y is a function of another variable x if y depends upon x.

We can express such a relationship by means of a formula.

Example

The area A of the circle is a function of its radius r:

$$A = \pi r^2$$

Function of one variable

There is no need of a formula to convey that one variable is a function of the other.

A table can also show the relationship.

Quarter	13Q1	13Q2	13Q3	13Q4	14Q1	14Q2	14Q3	14Q4
Consumption	1917.5	1924.9	1943.3	1946.0	1958.6	1973.4	1955.1	2008.2

Table 1: Final Consumption Expenditures in the EU, 2013Q1-2014Q4 (billions of euros)

This table defines consumption expenditures as a function of the calendar quarter.

Function of one variable

A graph can also illustrate the dependency between two variables. See the exhange rate EUR/USD during 2017:

Chapter 1: Functions and Continuity

Real Numbers

Notations

- $\forall \rightarrow$ for all
- $\exists \rightarrow$ exists
- $: \rightarrow$ such that
- $\in \rightarrow$ belongs
- $\subseteq \rightarrow$ contained in
- $\{\} \rightarrow$ set
- $\iff \rightarrow$ equivalently

Real Numbers

- Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Integers $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$
- Rationals $\mathbb{Q} = \{\frac{a}{b} \ : \ a \in \mathbb{Z}, b \in \mathbb{N}\}$
- Reals $\mathbb{R}$: $1, -6, \frac{-2}{3}, \pi, \sqrt{2} \in \mathbb{R}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Intervals

Given $a < b$:

- $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$
- $]a, b] = \{x \in \mathbb{R} \ : \ a < x \leq b\}$
- $[a, b[= \{x \in \mathbb{R} \ : \ a \leq x < b\}$
- $]a, b[= \{x \in \mathbb{R} \ : \ a < x < b\}$

Chapter 1: Functions and Continuity

Basic Definitions

The examples given in the introduction lead to the following general definition:

Definition: Function

A (real-valued) function f of a real variable x with domain $D \subset \mathbb{R}$ is a rule that assigns a unique real number y to each real number x in D. As x varies over the whole domain D, the set of all possible resulting values $f(x)$ is called the range of f.

Here rule means a formula, a table, a graph, etc.

We call x the independent variable, whereas y is called the dependent variable.

Example: The total dollar cost of producing x units of a product is $C(x) = 100x\sqrt{x} + 500$ for each nonnegative integer

The cost of producing 16 units is:

$$C(16) = 100 \cdot 16 \cdot \sqrt{16} + 500 = 6900$$

The cost of producing a units is $C(a) = 100a\sqrt{a} + 500$, and the cost of producing $a+1$ units is $C(a+1)$.

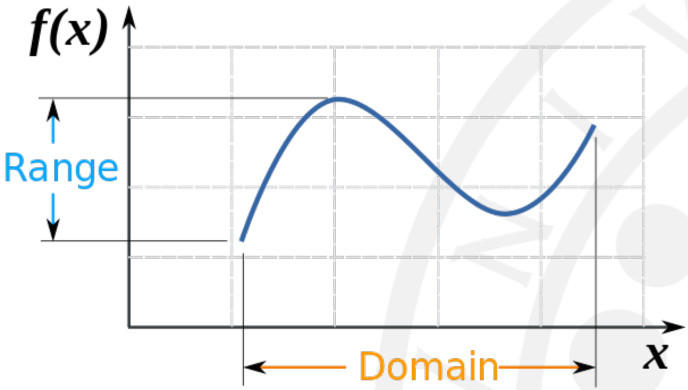
Thus the increase of cost from producing one additional unit is:

$$\begin{aligned} C(a+1) - C(a) &= 100(a+1)\sqrt{a+1} + 500 - (100a\sqrt{a} + 500) \\ &= 100\left[(a+1)\sqrt{a+1} - a\sqrt{a}\right] \end{aligned}$$

Domain and Range of a function

The domain of a function is the set of all the possible values of the independent variable.

The range of a function is teh set of all the possible values of the dependent variable.



Examples

Example: Find the domains of:

(a) $f(x) = \dfrac{1}{x+3}$ **(b)** $g(x) = \sqrt{2x+4}$

Example: Show that the number 4 belongs to the Range of the function $g(x) = \sqrt{2x+4}$**. Find the entire Range of** g**.**

Rules to compute domains

Let us consider two functions $p, q : D \to \mathbb{R}$, $\emptyset \neq D \subseteq \mathbb{R}$.

- $f(x) = \sqrt{p(x)} \to \mathrm{Dom}(f) = \{x \in D \ : \ p(x) \geq 0\}$
- $f(x) = \frac{p(x)}{q(x)} \to \mathrm{Dom}(f) = \{x \in D \ : \ q(x) \neq 0\}$
- $f(x) = p(x)^n \to \mathrm{Dom}(f) = D$, $n \in \mathbb{N}$

Increasing and decreasing functions

Definition: Increasing function

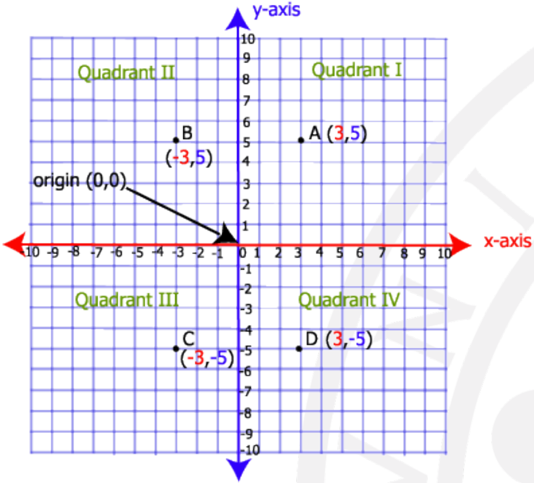
A function f is called increasing if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, and strictly increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

Definition: Decreasing function

A function f is called decreasing if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$, and strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Chapter 1: Functions and Continuity

Graphs of functions



y-axis
x-axis
Quadrant I
Quadrant II
Quadrant III
Quadrant IV
A (3,5)
B
(-3,5)
C
(-3,-5)
D (3,-5)
origin (0,0)

Graph

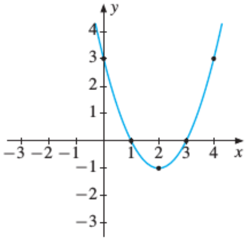
Definition: Graph

The graph of a function is the set of all ordered pairs $(x, f(x))$, where x belongs to the domain of f.

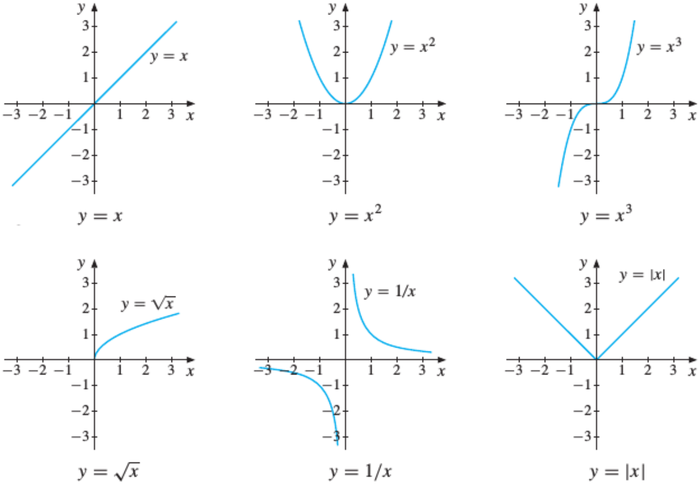
Example: Obtain the graph of the function:

(a) $f(x) = x^2 - 4x + 3$

x	0	1	2	3	4
$f(x) = x^2 - 4x + 3$	3	0	-1	0	3



Some Important graphs



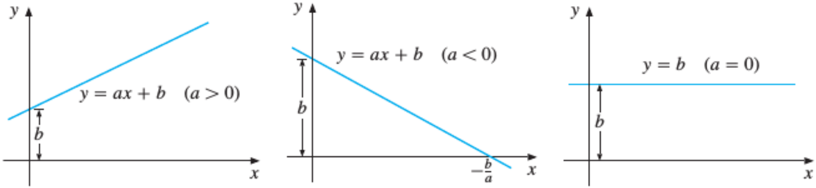
Chapter 1: Functions and Continuity

Linear Functions

Linear functions occur very often in economics. They are defined as:

$$y = f(x) = ax + b$$

with a (slope) and b constants.



Given (x_1, y_1) and (x_2, y_2) $(x_1 \neq x_2)$, the slope of the line from these points is given by

$$a = \frac{y_1 - y_2}{x_1 - x_2}$$

Example: Find and interpret the slopes and intercepts of:

(a) $C = 55.73x + 182100000$

Estimated cost function for the US Steel Corp. (1917-1938).
(C is the total cost in dollars per year, and x is the production of steel in tons per year).

(b) $q = -0.15p + 0.14$

Estimated annual demand function for rice in India for the period 1949-1964.
(p is price in Indian rupees, and q is consumption per person).

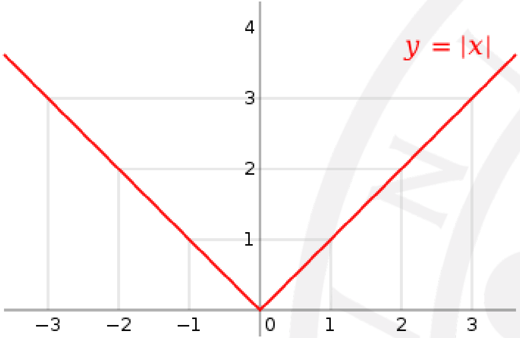
Chapter 1: Functions and Continuity

Absolute value

Absolute value

The absolute value is a piecewise linear function.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ +x & \text{if } x \geq 0 \end{cases}$$



Chapter 1: Functions and Continuity

Quadratic Functions

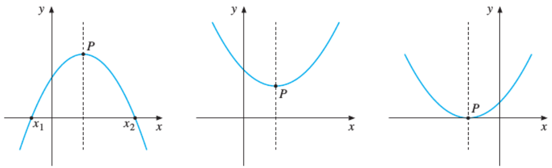
In general, linear functions are too simple for modelling economic phenomena with acceptable accuracy.

Indeed, many economic models involve functions that have a maximum or a minimum.

Some simple functions with this property are the general quadratic functions:

$$f(x) = ax^2 + bx + c \qquad (a,\ b,\ \text{and } c \text{ are constants, } a \neq 0)$$

The graph of $f(x) = ax^2 + bx + c$ is called a parabola.



(a) $a < 0,\ b^2 > 4ac$ (b) $a > 0,\ b^2 < 4ac$ (c) $a > 0,\ b^2 = 4ac$

Properties of the parabola

To characterize a parabola, we need to determine:

- The values of x for which $ax^2 + bx + c = 0$. We use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The coordinates of the maximum (or minimum) point of the parabola, called the vertex.

$$x = -\frac{b}{2a}$$

- If $a > 0$, it is said to be convex. If $a < 0$, it is said to be concave.

Chapter 1: Functions and Continuity

Polynomials

The function P defined for all x by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \,, \qquad (a_i \text{ are constants, } a_n \neq 0)$$

is called the general polynomial of degree n with coefficients $a_n, a_{n-1}, \ldots, a_0$.

One is interested in finding the number and location of the zeros/roots of $P(x)$, that is, the values of x such that $P(x) = 0$.

The equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

is called the general equation of degree n.

According to the fundamental theorem of algebra, every polynomial of degree n can be written as a product of polynomials of degree 1 or 2.

From this result it can be derived that the general equation of degree n has at most n (real) solutions, also called roots.

Factorization

Given a polynomial p of degree $n \in \mathbb{N}$, we can find $k \leq n$ polynomials $q_1, q_2, \ldots, q_k$ of degree 1 or 2 such that

$$p(x) = q_1(x) \cdot q_2(x) \cdot \ldots \cdot q_k(x)$$

Ruffini's method

Result:

The polynomial $P(x)$ has the factor $x - a \;\Leftrightarrow\; P(a) = 0$

Result:

Suppose that $a_n, a_{n-1}, \ldots, a_1, a_0$ are all integers. Then all possible integer roots of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

must be factors of the constant term a_0 .

Show that the polynomial $p(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$ has the roots $1, 1, -2, -2$.

Interpolating polynomial

Given $x_0 < x_1 < x_2 < \dots < x_n$, and $y_0, y_1, \dots, y_n \in \mathbb{R}$, there exists a unique polynomial of degree $n \in \mathbb{N}$ $p : \mathbb{R} \to \mathbb{R}$ such that

$$p(x_0) = y_0, p(x_1) = y_1, \dots, p(x_n) = y_n.$$

Such a polynomial is called the interpolating polynomial for the points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.

A rational function is a function $R(x) = P(x)/Q(x)$ that can be expressed as the ratio of two polynomials $P(x)$ and $Q(x)$.

This function is defined for all x where $Q(x) \neq 0$.

Example:

One of the simplest types of rational function is

$$R(x) = \frac{ax+b}{cx+d} \qquad (c \neq 0)$$

The graph of R is a hyperbola.

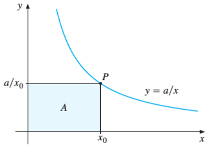


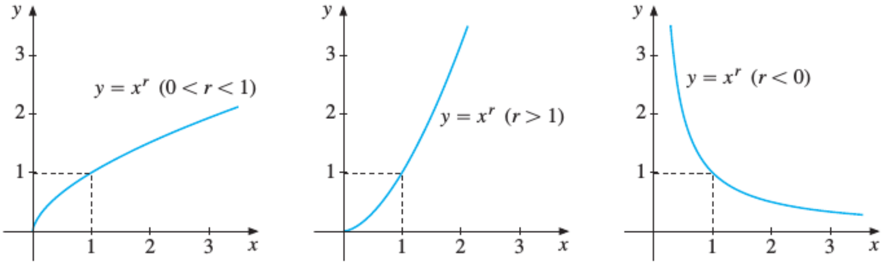
Figure 5 The area A is independent of P

Chapter 1: Functions and Continuity

Power functions

Consider the general power function f defined by the formula

$$f(x) = x^r \qquad (x > 0,\ r \text{ is constant})$$

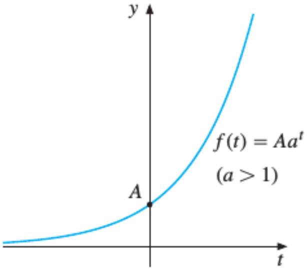


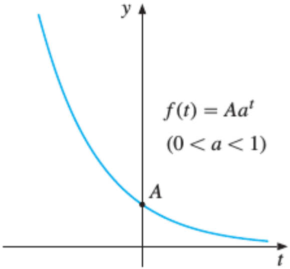
Chapter 1: Functions and Continuity

Exponential functions

A quantity that increases (or decreases) by a fixed factor per unit of time a is said to increase (or decrease) exponentially. This leads to the exponential function

$$f(t) = Aa^t \qquad (A, a \text{ are positive constants})$$





Each value a of $f(x) = a^x$ gives a different exponential function.

In mathematics, one particular value of a gives an exponential function that is far more important than all others.

This is the irrational number e, and its value to 15 decimals is:

$$e = 2.718281828459045\ldots$$

The corresponding exponential function

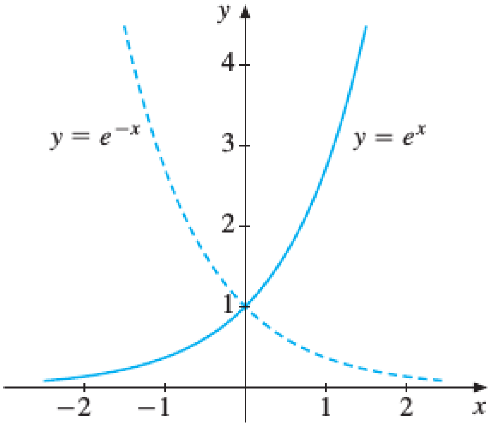
$$f(x) = e^x = exp(x)$$

is called the natural exponential function.

The usual rules applies for e^x. In particular:

$$(a) \quad e^s e^t = e^{s+t} \qquad (b) \quad \frac{e^s}{e^t} = e^{s-t} \qquad (c) \quad (e^s)^t = e^{st}$$

Graph of the Natural Exponential Function



Chapter 1: Functions and Continuity

Logarithm functions

Definition: Natural Logarithm

$ln(b) = x \iff e^x = b$

(a) $\ln(xy) = \ln x + \ln y$ (x and y positive)

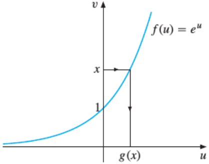
(b) $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$ (x and y positive)

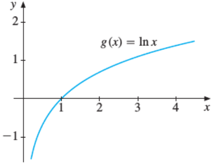
(c) $\ln\,(x^p) = p\,\ln(x)$ (x positive)

(d) $\ln\,(1) = 0$

(e) $\ln\,(e^x) = x$

(f) $\mathrm{Dom}(\ln) =]0, \infty[$





Chapter 1: Functions and Continuity

Application: Compound Interest

Let $r \in \mathbb{R}$ be the anual intererest, C_0 the seed money.

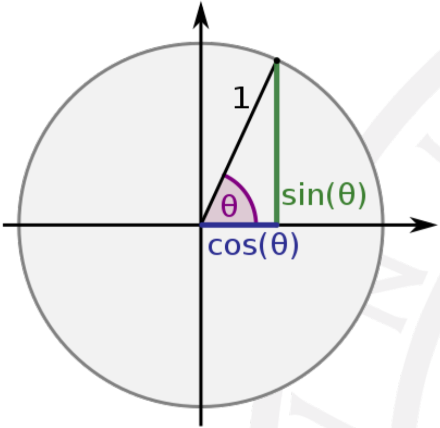
1. Compound Interest: $C(t) = C_0(1+r)^t, t \in \mathbb{N}$
2. Periodic Compound Interest: $C(t) = C_0(1+\frac{r}{m})^{mt}, t \in \mathbb{N}$, m is the number of periods.
3. Continuously Compound Interest: $C(t) = C_0 e^{rt}, t \in \mathbb{N}$.

Chapter 1: Functions and Continuity

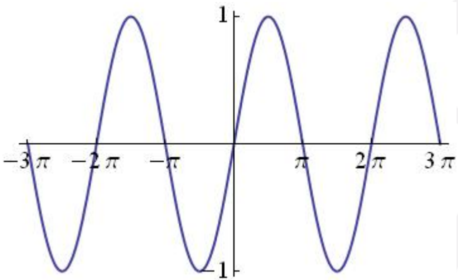
Trigonometric functions

Trigonometric functions

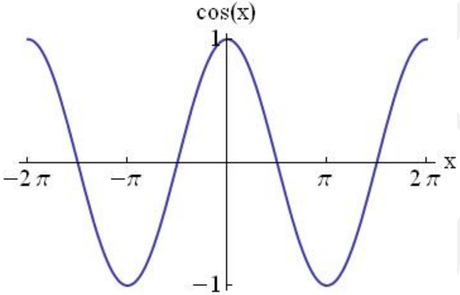
The basic sine and cosine functions are defined as follows:



Sine function



Cosine function



Properties of Sine and Cosine functions

- Their domain is $\mathbb{R}$
- Their range is $[-1, 1]$
- They are periodic with period 2π, i.e. $\sin(x+2\pi) = \sin(x)$ and $\cos(x+2\pi) = \cos(x)$
- $\cos^2(x) + \sin^2(x) = 1$
- Tangent function:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

  The domain is: $\mathbb{R} - \{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\}$
- If restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we find the inverses *arcsin*, *arccos*, *arctan*
- The argument x is measured in radians: π radians $= 180^\circ$

Chapter 1: Functions and Continuity

Limits and Continuity

Definition: Limit of a function

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real function, we say that f has limit L when x approaches $a \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists \delta > 0 \, : \, x \in D, \, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

We shall denote by

$$\lim_{x \to a} f(x) = L$$

In the definition of limit it is not necessary that f is defined in a, just that x can approach a.

Definition: Limit from below of a function

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real function, we say that f has limit L when x approaches $a \in \mathbb{R}$ from below if

$$\forall \epsilon > 0, \exists \delta > 0 \,:\, x \in D,\, 0 < |x - a| < \delta\,, x < a, \Rightarrow |f(x) - L| < \epsilon$$

We shall denote by

$$\lim_{x \to a^-} f(x) = L$$

In the definition of limit from below it is not necessary that f is defined in a, just that x can approach a from below.

Definition: Limit from above of a function

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real function, we say that f has limit L when x approaches $a \in \mathbb{R}$ from above if

$$\forall \epsilon > 0, \exists \delta > 0 \ : \ x \in D, \ 0 < |x - a| < \delta\,, x > a, \Rightarrow |f(x) - L| < \epsilon$$

We shall denote by

$$\lim_{x \to a^+} f(x) = L$$

In the definition of limit from above it is not necessary that f is definied in a, just that x can approach a from above.

Rules for Limits

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then

(a) $\lim_{x \to a}(f(x) \pm g(x)) = A \pm B$

(b) $\lim_{x \to a}(f(x)g(x)) = AB$

(c) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}$ (if $B \neq 0$))

(d) $\lim_{x \to a}(f(x))^r = A^r$ (if A^r is defined)

If the functions f and g are equal for all x close to a (but not necessarily at $x = a$), then $\lim_{x \to a} f(x) = lim_{x \to a} g(x)$ whenever either limit exists.

Definition: Limit at ∞ of a function

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a real function, with D not bounded from above and $a \in D$, we say that f has limit L when x tends to ∞ if

$$\forall \epsilon > 0, \exists M > 0 \, : \, x \in D, \, x > M \, \Rightarrow |f(x) - L| < \epsilon$$

We shall denote by

$$\lim_{x \rightarrow \infty} f(x) = L$$

Definition: Limit at $-\infty$ of a function

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a real function, with D not bounded from below and $a \in D$, we say that f has limit L when x tends to $-\infty$ if

$$\forall \epsilon > 0, \exists M > 0 \, : \, x \in D, \, x < -M \, \Rightarrow |f(x) - L| < \epsilon$$

We shall denote by

$$\lim_{x \rightarrow -\infty} f(x) = L$$

Limits at infinity

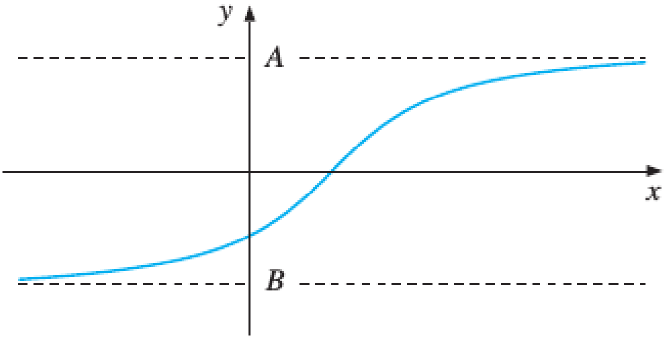


Figure 1: Limits at infinity represent horizontal asymptotes

The rules for limits also apply in the case of limits at infinity.

Infinite limits

In the definition of limit, we can allow for $L = \pm\infty$ and the definition still makes sense.

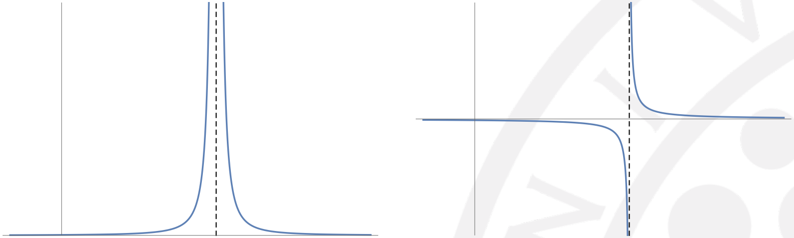


Figure 2: Infinite Limits represent vertical asymptotes

However, much more care is needed for the rules for infinite limits.

If both $f(x)$ and $g(x)$ tend to infinity at a, then both $f(x)+g(x)$ and $f(x)g(x)$ tend to infinity at a.

But, in general, we cannot say what are the limits of $f(x)-g(x)$ and $f(x)/g(x)$. The limits of these expressions will depend on how "fast" $f(x)$ and $g(x)$, respectively, tend to ∞ as x tends to a.

These ambiguities are known as indeterminate forms and there are more of them:

$$\infty-\infty\,,\ \frac{0}{0}\,,\ \frac{\infty}{\infty}\,,\ \infty\cdot 0\,,\ 1^{\infty}\,,\ 0^{0}\,,\ 0^{\infty}\,,\ \infty^{0}.$$

Rules for dealing with indeterminate forms

- If $x \to \infty$, then $e^x >> x^n >> ln(x)$
- If $x \to \infty$, then $x^n >> x^{n-1}$
- If $f(x) \to 1, g(x) \to \infty$ when $x \to a$, then

$$lim_{x \to a} f(x)^{g(x)} = e^{lim_{x \to a} g(x)(f(x)-1)}$$

Definition: Continuity

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real function and $a \in D$, we say that f is continuous at a if $\lim_{x\to a} = f(a)$, i.e

$$\forall \epsilon > 0, \exists \delta > 0 \, : \, x \in D, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

In the definition of continuity it is necessary that f is defined in a.

Result:

A function $f : D \to \mathbb{R}$ is continuous in $a \in D$ if and only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Properties of Continuous Functions

If f and g are continuous at a, then

(a) $f+g$ and $f-g$ are continuous at a

(b) fg and f/g (if $g(a)\neq 0$) are continuous at a

(c) $(f(x))^r$ is continuous at a (if $(f(a))^r$ is defined)

(d) If f is continuous and has an inverse on the interval I, then its inverse f^{-1} is continuous on $f(I)$.

If f is continuous at a and g is continuous at $f(a)$, then the composite function $g(f(x))$ is continuous at a.

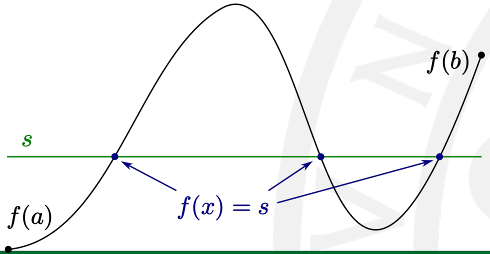
Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, multiplication, division (except by zero), and composition, is continuous at all points where it is defined.

Intermediate value Theorem

If $f : [a, b] \to \mathbb{R}$ is a continuous function, then it takes on any given value between $f(a)$ and $f(b)$.

Bolzano's Theorem

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval.



Chapter 2: Derivation

Chapter 2: Derivation

Introduction

Introduction

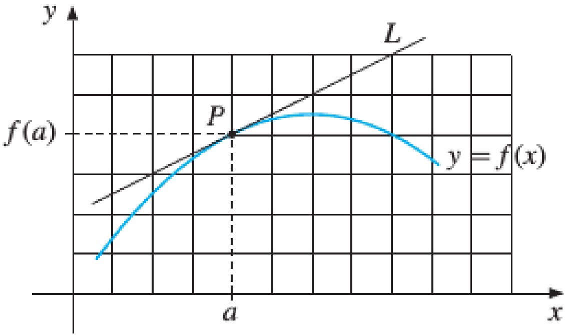
The concept of differentiation of a function appears in all areas of pure and applied mathematics, physics, chemistry, etc.

Also in economic analysis differentiation of functions appears, as the rate of change of functions, and allows for instance to estimate the future behavior of quantities like demand, supply, cost, production, consumption, etc.

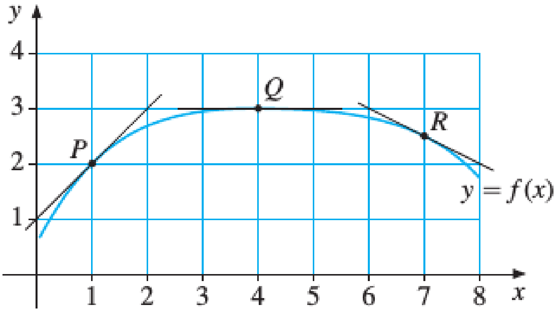
In this chapter we shall study differentiation of functions of one variable, illustrating the discussion with some important economic examples.

Slope of a curve

For a curve, the slope depends in general of the point. We can define the slope of a curve at a particular point as the slope of the tangent to the curve at that point. That is, as the slope of the straight line which just touches the curve at that point.



The slope of the tangent to the graph at $(a, f(a))$ is called the derivative of $f(x)$ at $x = a$, and we denote this number by $f'(a)$.



Chapter 2: Derivation

Derivatives

Consider a point P on a curve in the xy-plane. Take another point Q on the curve. The straight line through P and Q is called a secant. If we keep P fixed, but let Q move along the curve toward P, then the secant will rotate around P, as indicated in the Figure. The limiting straight line PT toward which the secant tends is called the tangent (line) to the curve at P.

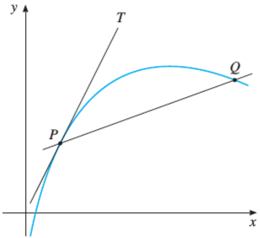


Figure 1

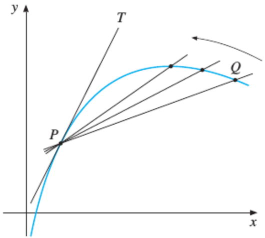
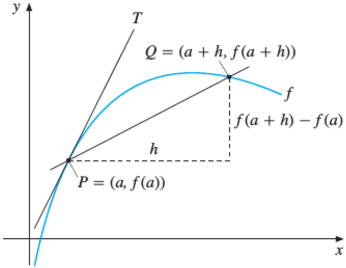


Figure 2

Newton quotient

Suppose the x-coordinate of Q is $a+h$, with $h \neq 0$. Then the y-coordinate of Q is $f(a+h)$. The slope m_{PQ} of the secant PQ is therefore:

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$



When Q moves toward P along the graph of f, the x-coordinate of Q, which is $a+h$, must tend to a, and so h tends to 0. Simultaneously, the secant PQ tends to the tangent to the graph at P. This suggests that we ought to define the slope of the tangent at P as the number that m_{PQ} approaches as h tends to 0.

Definition: Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition of the tangent

The equation for the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is

Definition: Equation of the tangent

$$y - f(a) = f'(a)(x - a)$$

Example

Compute $f'(a)$ when $f(x) = x^2$

Notation for the derivative

If $f(x) = x^2$, then for every a we have $f'(a) = 2a$. We frequently use x as the symbol for a quantity that can take any value, so we write $f'(x) = 2x$.

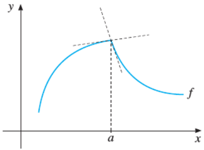
Several other forms of notation for the derivative are often used in mathematics and its applications. One of them is called the differential notation:

If $y = f(x)$, then in place of $f'(x)$, we write

$$\frac{dy}{dx} = dy/dx\,, \qquad \frac{df(x)}{dx} = df(x)/dx\,, \qquad \frac{d}{dx}f(x)$$

If the independent variable is t, derivatives are usually denoted with a dot: $\dot{f}(t)$

A function can be continuous at a point without being differentiable at that point.



Result:

If f is differentiable at $x = a$, then f is continuous at $x = a$.

Result:

If f is differentiable at $x = a$, then $lim_{x \to a-} f'(x) = lim_{x \to a+} f'(x)$.

Chapter 2: Derivation

Rules for differentiation

Differentiation

The derivative of a function was defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If the limit exists, we say that f is differentiable at x.

The process of finding the derivative of a function is called differentiation.

It is useful to think of this as an operation that transforms one function f into a new function f'. Instead of computing derivatives using the definition, we shall obtain general rules for finding the derivative of most differentiable functions specified by a formula.

Derivative of a constant

If f is a constant function, then its derivative is 0:

$$f(x) = A \quad \Rightarrow \quad f'(x) = 0$$

Additive constants disappear

$g(x) = A + f(x) \quad \Rightarrow \quad g'(x) = f'(x)$

Multiplicative constants are preserved

$g(x) = Af(x) \quad \Rightarrow \quad g'(x) = Af'(x)$

Differentiation of classical functions

Power rule

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

Exponential

$$f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x$$

Logarithm

$$f(x) = log(x) \quad \Rightarrow \quad f'(x) = \frac{1}{x}$$

Trigonometric

- $f(x) = sin(x) \quad \Rightarrow \quad f'(x) = cos(x),$
- $f(x) = cos(x) \quad \Rightarrow \quad f'(x) = -sin(x),$
- $f(x) = tan(x) \quad \Rightarrow \quad f'(x) = \frac{1}{cos^2(x)}$

Sums, Products, and Quotients

Differentiation of Sums and Differences

If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, and

$$F(x) = f(x) \pm g(x) \Rightarrow F'(x) = f'(x) \pm g'(x)$$

Differentiation of a product

If both f and g are differentiable at the point x, then so is $F = f \cdot g$, and

$$F(x) = f(x) \cdot g(x) \quad \Rightarrow \quad F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Differentiation of a quotient

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \quad \Rightarrow \quad F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Let f, g be functions such that $R_g \subset D_f$. Then the composite function $f \circ g$ is such that $(f \circ g)(x) = f(g(x))$.

Chain rule

If f and g are differentiable function such that $R_g \subset D_f$, then $f \circ g$ is differentiable and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Another way of stating the Chain rule is:

If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x and:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Chapter 2: Derivation

Derivatives and monotonicity

Assume f is defined on an interval I, and that x_1 and x_2 are in I:

Definition: Increasing function

f is called increasing in I if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, and strictly increasing in I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

Definition: Decreasing function

f is called decreasing in I if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$, and strictly decreasing in I if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Since increasing functions have positive (or zero) slope, whereas decreasing functions have negative (or zero) slope, we have:

$f'(x) \geq 0$ for all x in the interval $I \Leftrightarrow f$ is increasing in I

$f'(x) \leq 0$ for all x in the interval $I \Leftrightarrow f$ is decreasing in I

$f'(x) = 0$ for all x in the interval $I \Leftrightarrow f$ is constant in I

Strictly Increasing and Decreasing functions in terms of derivatives

$f'(x) > 0$ for all x in the interval $I \Rightarrow f$ is strictly increasing in I

$f'(x) < 0$ for all x in the interval $I \Rightarrow f$ is strictly decreasing in I

The implications cannot be reversed, in general.

Example

$f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

Chapter 2: Derivation

Derivatives and Rates of Change

Suppose a quantity y is related to a quantity x by $y = f(x)$.

If x has the value a, then the value of the function is $f(a)$.

Suppose that a is changed to $a + h$.

The new value of y is $f(a + h)$, and the change in the value of the function when x is changed from a to $a + h$ is $f(a + h) - f(a)$.

The change in y per unit change in x is the average rate of change of f over the interval from a to $a + h$. It is equal to

$$\frac{f(a+h) - f(a)}{h}$$

Taking the limit as h tends to 0 gives the derivative of f at a, $f'(a)$, which we interpret as the instantaneous rate of change of f at a.

Relative rate of Change

The relative rate of change of f at a is $\frac{f'(a)}{f(a)}$

Relative rate of change is also called proportional rate of change.

It is usually quoted in percentage per unit of time.

Rate of investment

Let $K(t)$ be the capital stock in an economy at time t. The rate of change $\dot{K}(t)$ of $K(t)$ is called the rate of investment at time t. It is usually denoted by $I(t)$, so

$$\dot{K}(t) = I(t)$$

Marginal cost

Let $C(x)$ denote the cost of producing x units. The derivative $C'(x) = \lim_{h\to 0} \frac{C(x+h)-C(x)}{h}$ at x is called the marginal cost at x.

When h is small in absolute value, the incremental cost of producing h units of extra output is $C(x+h) - C(x) \approx hC'(x)$.

For $h = 1$ marginal cost is approximately the incremental cost, $C'(x) \approx C(x+h) - C(x)$, that is, the additional cost of producing one more unit than x.

Chapter 2: Derivation

Higher-order derivatives

The derivative f' of a function f is often called the first derivative of f.

If f' is also differentiable, then we can differentiate f' in turn. The result $(f')'$ is called the second derivative, written more concisely as f''.

We use $f''(x)$ to denote the second derivative of f evaluated at the particular point x.

The second derivative can also be denotes as:

$$f''(x) = \frac{d^2 f(x)}{dx^2} \qquad \text{or} \qquad y'' = \frac{d^2 y}{dx^2}$$

nth-Order Derivatives

If $y = f(x)$, the derivative of $y'' = f''(x)$ is called the third derivative, customarily denoted by $y''' = f'''(x)$.

It is notationally cumbersome to continue using more and more primes to indicate repeated differentiation, so the fourth derivative is usually denoted by $y^{(4)} = f^{(4)}(x)$, os as $\frac{d^4 y}{dx^4}$.

In general

$$y^{(n)} = f^{(n)}(x) \qquad \text{or} \qquad \frac{d^n y}{dx^n}$$

denoted the n-th derivative of f at x.

Chapter 2: Derivation

Convexity

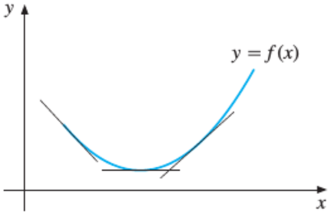
Convex and Concave Functions

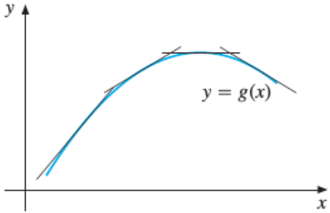
Recall how the sign of the first derivative determines whether a function is increasing or decreasing on an interval I.

The second derivative $f''(x)$ is the derivative of $f'(x)$. Hence:

$f''(x) \geq 0$ on $I \quad \Leftrightarrow \quad f'$ is increasing on I

$f''(x) \leq 0$ on $I \quad \Leftrightarrow \quad f'$ is decreasing on I





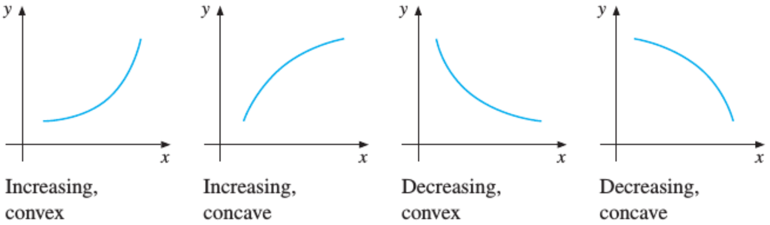
Convex and Concave Functions

Suppose that f is continuous in the interval I and twice differentiable in the interior of I.

Then we can introduce the following definitions:

f is convex on $I \quad \Leftrightarrow \quad f''(x) \geq 0$ for all x in I

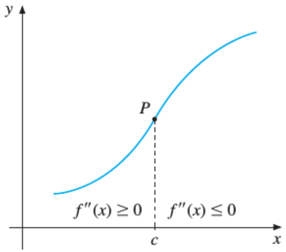
f is concave on $I \quad \Leftrightarrow \quad f''(x) \leq 0$ for all x in I



Points at which a function changes from being convex to being concave, or vice versa, are called inflection points. For twice differentiable functions they can be defined this way:

Inflection Point

Let $f : [a, b] \to \mathbb{R}$ be a twice differentible function. The point $c \in]a, b[$ is called an inflection point for the function f if there exists $\delta > 0$ such that $f''(x)f''(y) < 0$ $\forall x \in]c - \delta, c[, y \in]c, c + \delta[$.



Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function with a continuous second derivative and let $c \in]a, b[$.

- If c is an inflection point for f, then $f''(c) = 0$,
- If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f

Chapter 2: Derivation

Optimization

Relative and Absolute Extremes

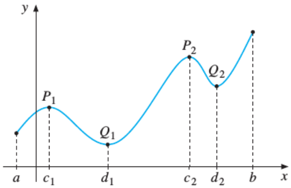
Consider a function $f : [a, b] \to \mathbb{R}$.

Relative/Local extremes

A point $c \in [a, b]$ is a relative/local maximum (minimum) if there exists $\delta > 0$ such that $f(c) \geq f(x)$ $(f(c) \leq f(x))$ $\forall x \in [c - \delta, c + \delta] \cap [a, b]$.

Absolute extremes

A point $c \in [a, b]$ is an absolute maximum (minimum) if $f(c) \geq f(x)$ $(f(c) \leq f(x))$ $\forall x \in [a, b]$.



If inequalities are strict we call them strict maximum or strict minimum points.

Stationary points

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. $c \in [a, b]$ is a stationary point if $f'(c) = 0$.

First-derivative test for Maximum/Minimum

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Let us consider a stationary point $c \in [a, b]$. Then:

- If there exists $\delta > 0$ such that $f'(x) \geq 0$ for $x \in]c - \delta, c[$ and $f'(x) \leq 0$ for $x \in]c, c + \delta[$, then c is a relative maximum point for f.
- If there exists $\delta > 0$ such that $f'(x) \leq 0$ for $x \in]c - \delta, c[$ and $f'(x) \geq 0$ for $x \in]c, c + \delta[$, then c is a relative minimum point for f.

Second-Derivative Tests for Maximum/Minimum

Theorem

Let f be a twice differentiable function in an interval $[a, b]$, and let $c \in]a, b[$. Then:

(a) If $f'(c) = 0$ and $f''(c) < 0 \Rightarrow x = c$ is a strict local maximum point.

(b) If $f'(c) = 0$ and $f''(c) > 0 \Rightarrow x = c$ is a strict local minimum point.

(c) If $f'(c) = 0$ and $f''(c) = 0 \Rightarrow$?

Problem: Find the maximum and minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$.

Solution:

(I) Find all stationary points of f in (a, b) – that is, find all points x in (a, b) that satisfy the equation $f'(x) = 0$.

(II) Evaluate f at the end points a and b of the interval and also at all stationary points.

(III) The largest function value found in (II) is the maximum value, and the smallest function value is the minimum value of f in $[a, b]$.

Theorem: The Extreme Value Theorem

Suppose that f is a continuous function over a closed and bounded interval $[a, b]$. Then there exist a point d in $[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum, so that

$$f(d) \leq f(x) \leq f(c) \qquad \text{for all} x \in [a, b]$$

Theorem: Extreme Points for Concave and Convex Functions

Suppose f is a concave (convex) function in an interval I. If c is a stationary point for f in the interior of I, then c is a maximum (minimum) point for f in I.

Chapter 2: Derivation

L'Hôpital's Rule

L'Hôpital's Rule

L'Hôpital's Rule allows to compute indeterminate forms of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Theorem: L'Hôpital's Rule

Suppose that f and g are differentiable in an interval I that contains a, except possibly at a, and suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} |g(x)| = \infty$. If $g'(x) \neq 0$ for all $x \neq a$ in I, and if $\lim_{x\to a} f'(x)/g'(x) = L$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

This is true whether L is finite or $\pm\infty$.

L'Hôpital's Rule also applies when $a = \pm\infty$ or with one-sided limits.

Chapter 3: Integration

Chapter 3: Integration

Introduction

The concept of integration is two-fold. In one side, its appears as the inverse procedure to differentiation (antiderivative, primitive or indefinite integral), and on the other side, it appears in relation to the computation of areas of plane figures.

Chapter 3: Integration

Indefinite Integral

Definition: Primitive

If $F(x)$ is such that $F'(x) = f(x)$, then $F(x)$ is called a primitive of $f(x)$.

Since the derivative of a constant is zero, $F(x)$ is determined up to a constant. We shall call Indefinite Integral to all the possible primitives of a function:

Definition: Indefinite Integral

If $F'(x) = f(x)$, then the indefinite integral is defined as:

$$\int f(x)dx = F(x) + C$$

where C in a constant.

Indefinite Integral

The symbol $\int$ is the integral sign, and $f(x)$ is the integrand.

We write dx to indicate that x is the variable of integration.

C is a constant of integration.

Differentiating each side shows directly that

$$\frac{d}{dx}\int f(x)dx = f(x)$$

or

$$\int F'(x)dx = F(x) + C$$

Thus, integration and differentiation cancel each other out.

Chapter 3: Integration

Elemental Integration Rules

Constants

There are some important integration formulas which follow immediately from the corresponding rules for differentiation.

Integral of zero

$$\int 0dx = C$$

Integral of a constant

$$\int kdx = kx + C$$

Integral of a power

Let $a \neq -1$ be a fixed number. Then:

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

For the case $a = -1$ we have:

Integral of the logarithm

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

For the exponential function we have:

Integral of the exponential

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C \qquad (a \neq 0)$$

Trigonometric functions

Sine function

$$\int sin(x)dx = -cos(x) + C$$

Cosine function

$$\int cos(x)dx = sin(x) + C$$

Tangent function

$$\int tan(x)dx = 1 + tan^2(x) + C$$

Arctangent function

$$\int \frac{1}{1+x^2}dx = Arctan(x) + C$$

Linearity

The two differentiation rules $(aF(x))' = aF'(x)$ and $(F(x)+G(x))' = F'(x)+G'(x)$ immediately imply the following integration rules

$$\int af(x)\,dx = a\int f(x)\,dx \qquad a \text{ is a constant}$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

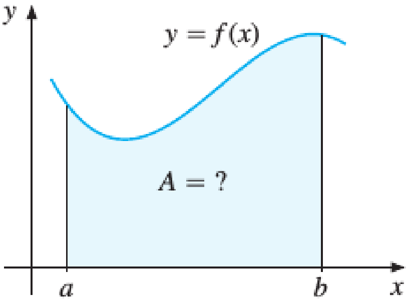
Chapter 3: Integration

Areas and Definite Integral

Area and Definite Integral

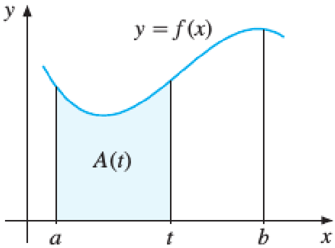
This section will show how the concept of the integral can be used to calculate the area of many plane regions.

The problem to be considered and solved in this section is: How do we compute the area A under the graph of a continuous and nonnegative function f over the interval $[a, b]$?



Area and Definite Integral

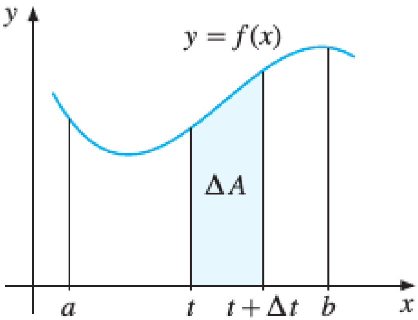
Let t be an arbitrary point in $[a, b]$, and let $A(t)$ denote the area under the curve $y = f(x)$ over the interval $[a, t]$:

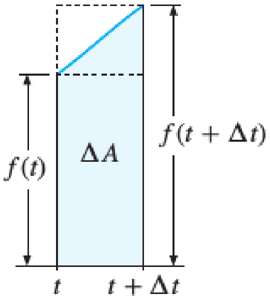


Clearly, $A(a) = 0$, because there is no area from a to a

On the other hand, the total area $A = A(b)$. It is obvious that, because f is always positive, $A(t)$ increases as t increases.

Suppose we increase t by a positive amount Δt. Then $A(t + \Delta t)$ is the area under the curve $y = f(x)$ over the interval $[a, t + \Delta t]$. Hence, $A(t + \Delta t) - A(t)$ is the area ΔA under the curve over the interval $[t, t + \Delta t]$.





The area ΔA cannot be larger than the area of the rectangle with base Δt and height $f(t+\Delta t)$, nor smaller than the area of the rectangle with base Δt and height $f(t)$. Hence, for all $\Delta t > 0$, one has

$$f(t)\Delta t \leq A(t+\Delta t) - A(t) \leq f(t+\Delta t)\Delta t$$

Since $\Delta t > 0$ this implies

$$f(t) \leq \frac{A(t+\Delta t) - A(t)}{\Delta t} \leq f(t+\Delta t)$$

When $\Delta t \to 0$:

$$A'(t) = f(t) \text{ for all } t \in (a, b)$$

This is also true in the case where $f(t)$ is not increasing (it is enough that f is continuous).

Let $F(x)$ be an arbitrary primitive of $f(x)$. Then $A(x) = F(x) + C$. Since $A(a) = 0$, then $0 = A(a) = F(a) + C$, so $C = -F(a)$. Therefore

$$A(x) = F(x) - F(a) \qquad \text{where } F(x) = \int f(x)\,dx$$

Definition: Definite Integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any primitive of f over an interval containing both a and b.

Area of a negative function

If $f(x) \geq 0$ over $[a, b]$, then

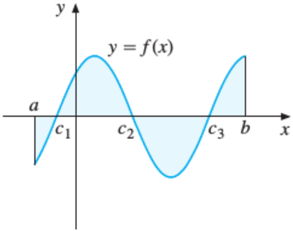
$$\int_a^b f(x)\,dx$$

is the area below the graph of f over $[a, b]$.

If f is defined in $[a, b]$ and $f(x) \leq 0$ for all x in $[a, b]$, then the graph of f, the x-axis and the lines $x = a$ and $x = b$ still enclose an area. This area is $-\int_a^b f(x)\,dx$, and the minus sign is because the definite integral is negative and the area must be positive (or zero).

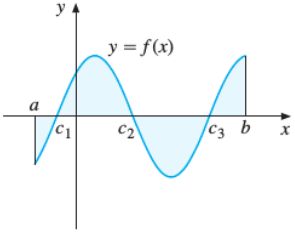
Area for arbitrary functions

Suppose the function f is defined and continuous in $[a, b]$, and that it is positive in some subintervals, and negative in others:



Let c_1 , c_2 , c_3 denote three roots of the equation $f(x) = 0$. The definite integral $\int_a^b f(x)\,dx$ is the sum of the two shaded areas above the x-axis, minus the sum of the two shades areas below the x-axis.

Area for arbitrary functions



The total area bounded by the graph of f, the x-axis and the lines $x = a$ and $x = b$ is calculated by computing the positive areas in each subinterval $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$, and $[c_3, b]$ and then adding these areas.

The total area is then:

$$-\int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx - \int_{c_2}^{c_3} f(x)\,dx + \int_{c_3}^{b} f(x)\,dx = \int_a^b |f(x)|\,dx$$

Properties of definite integrals

If f is a continuous function in an interval that contains a, b, and c, then:

$$\begin{aligned}
\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx \\
\int_a^a f(x)\,dx &= 0 \\
\int_a^b \alpha f(x)\,dx &= \alpha \int_a^b f(x)\,dx \qquad (\alpha \in \mathbb{R}) \\
\int_a^b [\alpha f(x) + \beta g(x)]\,dx &= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \quad (\alpha, \beta \in \mathbb{R}) \\
\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx
\end{aligned}$$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and consider the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)dt$. Then,

$$F'(x) = f(x) \; \forall x \in]a, b[$$

Chapter 3: Integration

Economic Applications

Economic Applications

We motivated the definite integral as a tool for computing the area under a curve.

However, the integral has many other important interpretations. In statistics, many important probability distributions are expressed as integrals of continuous probability density functions.

Let's see some examples showing the importance of integrals in economics.

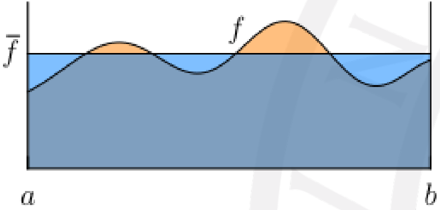
Average value of a function

Definition:Average value of a function

Given the function $f : [a, b] \to \mathbb{R}$, we define the average value of f as the real number

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)dx$$

The average value $\bar{f}$ represents the constant function on $[a, b]$ such that the enclosed area under its graph and the X axis between a and b has the same area as the graph of f.



Variable Continuously Compound Interest

In certain cases, the interest we obtain in a bank account, or the interest we pay in a mortgage loan is variable in time, $I(t)$. The simplest case is when the interest is piecewise constant.

For the caso of variable continuously compound interest, it is possible to give an expression for the capital as a function of time:

$$C(t) = C_0 e^{\frac{1}{100}\int_{t_0}^{t} I(\tau)d\tau}$$

where C_0 is the original capital at t_0 (usually we take$t_0 = 0$).

Chapter 3: Integration

Advanced Integration Rules

In general, because the derivative of a product is not the product of the derivatives, the integral of a product is not the product of the integrals.

The product rule for differentiation states that:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Now take the indefinite integral of each side

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Rearranging this last equation yields the following formula:

Integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

The point is to choose f and g so that it is easier to find $\int f'(x)g(x)dx$ than is to find $\int f(x)g'(x)dx$. We follow the following rule to choose f:

ALPES

1. A - Arcsine, Arccosine, Arctan,. . . ,
2. L - Logarithm,
3. P - Polynomials,
4. E - Exponential,
5. S - Sine, Cosine,...

There is a corresponding result for definite integrals.

From the definition of the definite integral and the product rule for differentiation, we have:

Integration by parts Definite Integrals:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Integration by substitution

From the chain rule for the differentiation of the composite function $f \circ g$, the corresponding rule can be derived for integration:

Immediate Integrals

Let f and g be functions such that $R_g \subset D_f$, with g differentiable and f continuous. Then

$$\int f'(g(x))g'(x)dx = f(g(x))$$

Change of variable

Let $\phi : [a, b] \to \mathbb{R}$ be a continuously differentiable bijection with continuously differentiable inverse. Then, for $f : \phi([a, b]) \to \mathbb{R}$ continuous, we have:

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)dt$$

Integration of Rational Functions

The integration of rational functions of the form $\frac{P(x)}{Q(x)}$ requires the use of two techniques.

First, if the degree of $P(x)$ is larger or equal to that of $Q(x)$, we apply polynomial division to obtain:

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

with the degree of $r(x)$ smaller than that of $Q(x)$.

Second, apply Partial Fraction Expansion to expand $\frac{r(x)}{Q(x)}$ into a sum of simpler fractions.

Chapter 4: Matrices and Systems of Equations

Chapter 4: Matrices and Systems of Equations

Introduction

Introduction

Most mathematical models used by economists ultimately involve a system of several equations.

If these equations are all linear, the study of such systems belongs to an area of mathematics called Linear Algebra.

The analysis and even the comprehension of systems of linear equations becomes much easier if we use some key mathematical concepts such as matrices, vectors, and determinants.

We consider m equations in n unknowns, where m may be greater than, equal to, or less than n. If the unknowns are denoted by $x_1, \ldots, x_n$, we usually write such a system in the form:

$$\begin{array}{ccccccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & +a_{1j}x_j + & \cdots & +a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & +a_{2j}x_j + & \cdots & +a_{2n}x_n & = & b_2 \\
 & & \vdots & & & & \vdots \\
a_{i1}x_1 + a_{i2}x_2 + & \cdots & +a_{ij}x_j + & \cdots & +a_{in}x_n & = & b_i \\
 & & \vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + & \cdots & +a_{mj}x_j + & \cdots & +a_{mn}x_n & = & b_m
\end{array}$$

Here a_{11}, a_{12}, . . . , a_{mn} are called the coefficients of the system, and b_1, . . . , b_m are called the right-hand sides.

a_{ij} is the coefficient in the i-th equation of the j-th unknown (x_j)

A solution of the linear system is an ordered set or list of numbers s_1, s_2, ..., s_n that satisfies all the equations simultaneously when we put $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$.

Usually, a solution is written as $(s_1, s_2, \ldots, s_n)$.

If the linear system has at least one solution, it is said to be consistent. When the system has no solution, it is said to be inconsistent.

Chapter 4: Matrices and Systems of Equations

Matrices and Matrix Operations

A matrix is a rectangular array of numbers.

When there are m rows and n columns in the array, we have an m–by–n matrix (written as $m \times n$).

We usually denote a matrix with bold capital letters such as $\mathbf{A}$, $\mathbf{B}$, and so on.

In general, an $m \times n$ matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The matrix $\mathbf{A}$ is said to have order $m \times n$. The mn numbers that constitute $\mathbf{A}$ are called its elements or entries.

In particular, a_{ij} denotes the element in the i-th row and the j-th column.

For brevity, the $m \times n$ matrix is often expressed as $(a_{ij})_{m \times n}$, or more simply as (a_{ij}), if the order $m \times n$ is either obvious or unimportant.

A matrix with either only one row or only one column is called a vector.

It is usual to distinguish between a row vector, which has only one row, and a column vector, which has only one column.

It is usual to denote row or column vectors by small bold letters like **x** or **y** rather than capital letters.

An important case is when $m = n$, so that the matrix has the same number of columns as rows, it is called a square matrix of order n. If $\mathbf{A} = (a_{ij})_{n\times n}$, then the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute the main diagonal that runs from the top left (a_{11}) to the bottom right (a_{nn}).

Consider the general linear system:

$$\begin{array}{ccccccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & +a_{1j}x_j + & \cdots & +a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & +a_{2j}x_j + & \cdots & +a_{2n}x_n & = & b_2 \\
 & & \vdots & & & & \vdots \\
a_{i1}x_1 + a_{i2}x_2 + & \cdots & +a_{ij}x_j + & \cdots & +a_{in}x_n & = & b_i \\
 & & \vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + & \cdots & +a_{mj}x_j + & \cdots & +a_{mn}x_n & = & b_m
\end{array}$$

of m equations in the n unknown variables x_j ($j = 1, 2, \ldots, n$).

It is natural to represent the coefficients of the n unknowns by the $m \times n$ matrix $\mathbf{A}$ that is arranged as:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Example

The coefficient matrix of the system of equations:

$$\begin{array}{rcr} 3x_1 - 2x_2 + 6x_3 & = & 5 \\ 5x_1 + x_2 + 2x_3 & = & -2 \end{array} \quad \text{is} \quad \begin{pmatrix} 3 & -2 & 6 \\ 5 & 1 & 2 \end{pmatrix}$$

One can also represent the numbers b_i $(i = 1, 2, \ldots, m)$ on the right-hand side of the system of equations by an $m \times 1$ matrix, or column vector, denoted by $\mathbf{b}$.

Example

The right-hand side of the system of equations:

$$\begin{array}{rcr} 3x_1 - 2x_2 + 6x_3 & = & 5 \\ 5x_1 + x_2 + 2x_3 & = & -2 \end{array} \quad \text{is} \quad \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

So far matrices have been regarded as just rectangular arrays of numbers that can be useful for storing information.

The real motivation for introducing matrices, however, is that there are useful rules for manipulating them that correspond (to some extent) with the familiar rules of ordinary algebra.

Equality of matrices:

If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ are both $m \times n$ matrices, then $\mathbf{A}$ and $\mathbf{B}$ are said to be equal, and we write $\mathbf{A} = \mathbf{B}$, provided that $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$. Thus, two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if they have the same order and if all their corresponding entries are equal. If $\mathbf{A}$ and $\mathbf{B}$ are not equal, then we write $\mathbf{A} \neq \mathbf{B}$.

Sum of matrices:

In general, if $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$ are two matrices of the same order, we define the sum of $\mathbf{A}$ and $\mathbf{B}$ as the $m \times n$ matrix $(a_{ij} + b_{ij})_{m \times n}$. Thus,

$$\mathbf{A} + \mathbf{B} = (a_{ij})_{m \times n} + (b_{ij})_{m \times n} = (a_{ij} + b_{ij})_{m \times n}$$

So we add two matrices of the same order by adding their corresponding entries.

Multiplication of matrices by scalars:

If α is a real number, we define $\alpha\mathbf{A}$ by:

$$\alpha\mathbf{A} = \alpha(a_{ij})_{m \times n} = (\alpha a_{ij})_{m \times n}$$

Thus, to multiply a matrix by a scalar, multiply each entry in the matrix by that scalar.

Example

Compute $\mathbf{A}+\mathbf{B}$, $3\mathbf{A}$, and $(-2)\mathbf{B}$, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -3 & -1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

The matrix $(-1)\mathbf{A}$ is usually denoted by $-\mathbf{A}$, and the difference between the two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, $\mathbf{A}-\mathbf{B}$, means the same as $\mathbf{A}+(-1)\mathbf{B}$.

With the definitions given earlier, it is easy to derive some useful rules. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be arbitrary $m \times n$ matrices, and let α and β be real numbers. Also, let $\mathbf{0}$ denote the $m \times n$ matrix consisting only of zeros, called the zero matrix.

Rules for Matrix Addition and Multiplication by Scalars

Rules for Matrix Addition and Multiplication by Scalars

(a) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

(b) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(c) $\mathbf{A}+\mathbf{0}=\mathbf{A}$

(d) $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$

(e) $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$

(f) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$

Each of these rules follows directly from the definitions and the corresponding rules for ordinary numbers.

Because of rule (a), there is no need to put parentheses in expressions like $\mathbf{A}+\mathbf{B}+\mathbf{C}$.

Note also that (e) implies that $\mathbf{A}+\mathbf{A}$ is equal to $2\mathbf{A}$.

Chapter 4: Matrices and Systems of Equations

Matrix Multiplication

The rules we just gave for adding or substracting matrices, and for multiplying a matrix by a scalar, should seem quite natural. The rule for matrix multiplication, however, is more subtle.

Matrix Multiplication

Suppose that $\mathbf{A} = (a_{ij})_{m\times n}$ and that $\mathbf{B} = (b_{ij})_{n\times p}$. Then the product $\mathbf{C} = \mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C} = (c_{ij})_{m\times p}$, whose element in the i-th row and the j-th column is the inner product

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ik} b_{kj} + \ldots + a_{in} b_{nj}$$

Note that to get c_{ij} we multiply each component a_{ir} in the i-th row of **A** by the corresponding component b_{rj} in the j-th column of **B**, then add all the products. One way of visualizing matrix multiplication is this:

$$\begin{pmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \boxed{a_{i1}} & \dots & a_{ik} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mk} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \dots & b_{kj} & \dots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \boxed{c_{ij}} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix}$$

The matrix product **AB** is defined only if the number of columns in **A** is equal to the number of rows in **B**. Also, if **A** and **B** are two matrices, then **AB** might be defined, even if **BA** is not.

For instance, if **A** is 6×3 and **B** is 3×5, then **AB** is defined (and is 6×5), whereas **BA** is not defined.

Example

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$. Compute the matrix product $\mathbf{AB}$. Is the product $\mathbf{BA}$ defined?

In the previous example, $\mathbf{AB}$ was defined but $\mathbf{BA}$ was not. Even in cases in which $\mathbf{AB}$ and $\mathbf{BA}$ are both defined, they are usually not equal.

We set out some rather obvious algebraic rules for matrix addition and multiplication by a scalar. Matrix multiplication is a more complicated operation, so we must carefully examine what rules apply. We have already noticed that the commutative law $\mathbf{AB} = \mathbf{BA}$ does NOT hold in general. The following three important rules are generally valid, however.

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices whose dimensions are such that the given operations are defined, and α is an arbitrary scalar, then:

Rules for Matrix Multiplication

(a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative law)

(b) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (left distributive law)

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ (right distributive law)

(d) $(\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B}) = \alpha\mathbf{AB}$

Systems of Equations in Matrix Form

The definition of matrix multiplication was introduced in order to allow systems of equations to be manipulated. Indeed, it turns out that we can write linear systems of equations very compactly by means of matrix multiplication. For instance, consider the system:

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$$

Now define $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Then we see that

$$\mathbf{Ax} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

So the original system is equivalent to the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

Consider the general linear system with m equations and n unknowns:

$$\begin{array}{ccccccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & +a_{1j}x_j + & \cdots & +a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & +a_{2j}x_j + & \cdots & +a_{2n}x_n & = & b_2 \\
 & & \vdots & & & & \vdots \\
a_{i1}x_1 + a_{i2}x_2 + & \cdots & +a_{ij}x_j + & \cdots & +a_{in}x_n & = & b_i \\
 & & \vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + & \cdots & +a_{mj}x_j + & \cdots & +a_{mn}x_n & = & b_m
\end{array}$$

Suppose we define

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Then the system of equations can be written as $\mathbf{Ax} = \mathbf{b}$.

If $\mathbf{A}$ is a square matrix, the associative law allows us to write $\mathbf{AA}$ as $\mathbf{A}^2$, and $\mathbf{AAA}$ as $\mathbf{A}^3$, and so on. In general,

$$\mathbf{A}^n = \mathbf{AA}\cdots\mathbf{A} \qquad (\mathbf{A} \text{ is repeated } n \text{ times})$$

Example:

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute $\mathbf{A}^2$, $\mathbf{A}^3$, and $\mathbf{A}^4$. Then guess the general form of $\mathbf{A}^n$.

The Identity Matrix

The identity matrix of order n, denoted by $\mathbf{I}_n$ (or often just by $\mathbf{I}$), is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

If $\mathbf{A}$ is any $m \times n$ matrix, it is easy to verify that $\mathbf{A}\mathbf{I}_n = \mathbf{A}$. Likewise, if $\mathbf{B}$ is any $n \times m$ matrix, then $\mathbf{I}_n\mathbf{B} = \mathbf{B}$. In particular,

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}\,, \qquad (\text{for every } n \times n \text{ matrix } \mathbf{A})$$

Chapter 4: Matrices and Systems of Equations

The Transpose

The Transpose

Consider any $m \times n$ matrix $\mathbf{A}$. The transpose of $\mathbf{A}$, denoted by $\mathbf{A}'$, is defined as the $n \times m$ matrix whose first column is the first row of $\mathbf{A}$, whose second column is the second row of $\mathbf{A}$, and so on. Thus,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

So we can write $\mathbf{A}' = (a'_{ij})$ where $a'_{ij} = a_{ji}$. The subscripts i and j have to be interchanged because the j-th row of $\mathbf{A}$ becomes the j-th column of $\mathbf{A}'$, whereas the i-th column of $\mathbf{A}$ becomes the i-th row of $\mathbf{A}'$.

Example:

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$. Find $\mathbf{A}'$ and $\mathbf{B}'$.

Rules for transposition

(a) $(\mathbf{A}')' = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$

(c) $(\alpha \mathbf{A})' = \alpha \mathbf{A}'$

(d) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

Square matrices with the property that they are symmetric about the main diagonal are called symmetric. For example,

$$\begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 5 \\ -1 & -3 & 2 \\ 5 & 2 & 8 \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

are all symmetric.

Symmetric matrices are characterized by the fact that they are equal to their own transposes:

$$\text{The matrix } \mathbf{A} \text{ is symmetric} \Leftrightarrow \mathbf{A} = \mathbf{A}'$$

Hence, matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is symmetric iff $a_{ij} = a_{ji}$ for all i, j.

Chapter 4: Matrices and Systems of Equations

Determinants

Determinants

We shall define the determinant of a square matrix recursively.

For a 1×1 matrix $\mathbf{A} = (a_{11})$, we define the determinant as $|\mathbf{A}| = a_{11}$.

For $n \times n$ matrix $\mathbf{A} = (a_{ij})_{n\times n}$ we define the determinant as:

(expansion by row i) $|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{ij}C_{ij} + \ldots + a_{in}C_{in}$

or

(expansion by column j) $|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{ij}C_{ij} + \ldots + a_{nj}C_{nj}$

The coefficients $C_{i1}, \ldots, C_{in}$ are the cofactors of the elements $a_{i1}, \ldots, a_{in}$, and $C_{1j}, \ldots, C_{nj}$ are the cofactors of the elements $a_{1j}, \ldots, a_{nj}$.

The cofactor C_{ij} is obtained by first deleting row i and column j, and then computing the determinant of the resulting submatrix of order $n-1$, which is called a minor. Finally, multiply the minor by the factor $(-1)^{i+j}$. This gives:

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & \boxed{a_{ij}} & a_{i,j+1} & \cdots & a_{in} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Examples

Determinant of order 2:

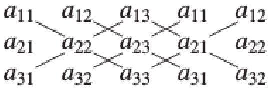
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of order 3:

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ &\quad - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

There is an alternative way of evaluating determinants of order 3 that many be convenient.

Write down the determinant twice, except that the last column in the second determinant should be omitted:



First, multiply along the three lines falling to the right, giving all these products a plus sign:

$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

Then multiply along the three lines rising to the right, giving all these products a minus sign:

$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Rules for Determinants

Let $\mathbf{A}$ be an $n \times n$ matrix. Then:

A. If the elements in a row (or column) of $\mathbf{A}$ are 0, then $|\mathbf{A}| = 0$.

B. $|\mathbf{A}'| = |\mathbf{A}|$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

C. If all the elements in a single row (or column) of $\mathbf{A}$ are multiplied by a number α, the determinant is multiplied by α.

D. If two rows (or two columns) of $\mathbf{A}$ are interchanged, the determinant changes sign.

E. If two rows (or columns) of $\mathbf{A}$ are proportional, then $|\mathbf{A}| = 0$.

F. The determinant of $\mathbf{A}$ is unchanged if a multiple of one row (or one column) is added to a different row (or column) of $\mathbf{A}$.

G. The determinant of the product of two $n \times n$ matrices A and B is the product of the determinants of each of the factors:
$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$

H. If α is a real number,
$|\alpha \mathbf{A}| = \alpha^n |\mathbf{A}|$

Chapter 4: Matrices and Systems of Equations

Inverse Matrix

Inverse Matrix

A square matrix $\mathbf{A}$ of order n is invertible or regular if there exists another square matrix $\mathbf{B}$ of order n such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

The matrix $\mathbf{B}$ is called the inverse matrix of $\mathbf{A}$ and it is denoted by $\mathbf{A}^{-1}$.

Example

Check that the inverse matrix of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two regular matrices of the same order. Then:

- $\mathbf{AB}$ is regular and we have that: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $\mathbf{A}'$ is regular and we have that: $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
- $|\mathbf{A}^{-1}| = \dfrac{1}{|\mathbf{A}|}$. In particular, $|\mathbf{A}| \neq 0$.
- If $\alpha \neq 0$, then $\alpha\mathbf{A}$ is regular and $(\alpha\mathbf{A})^{-1} = \dfrac{1}{\alpha}\mathbf{A}^{-1}$

The expression of the inverse matrix can be obtained using determinants. For that purpose let us define the adjoint matrix $Adj(\mathbf{A})$ of a square matrix $\mathbf{A}$ as:

$$Adj(\mathbf{A}) = (C_{ij})_{n\times n}$$

where C_{ij} are the cofactors previously defined (known also as adjoints).

We then have:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} Adj(\mathbf{A})'$$

Ejemplo

Compute the inverse matrix of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ using determinants.

Chapter 4: Matrices and Systems of Equations

Rank of a matrix

Definition: Linear Combination of vectors

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of m n-vectors. Denote by $\langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \rangle$ the set of vectors $\mathbf{w}$ that are obtained as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$:

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m$$

Definition: Linear dependence and independence of vectors

- Let $\mathbf{v}$ be a n-vector. We shall say that $\mathbf{v}$ is linearly independent if $\mathbf{v} \neq \mathbf{0}$. If $\mathbf{v} = \mathbf{0}$ we shall say that it is linearly dependent.
- Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of m n-vectors. We shall say that it is linearly dependent if some of them is a linear combination of the rest.
  If none of them is a linear combination of the rest we say that it is linearly independent.

Maximal number of linearly independent vectors

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of m n-vectors. Suppose that it is linearly dependent and that $\mathbf{v}_m$ is a linear combination of the rest. Then:

$$\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \rangle = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m-1} \rangle \equiv \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m-1}, \not{\mathbf{v}}_m \rangle$$

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m-1}\}$ is also linearly dependent, we repeat the process until we delete all vectors that can be written as a linear combination of the rest, in such a way that the remaining subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k < m$ is linearly independent. Then we have that:

$$\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \rangle = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \rangle \equiv \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \not{\mathbf{v}}_{k+1}, \ldots, \not{\mathbf{v}}_m \rangle$$

Maximal number of linearly independent vectors

Example:

Find the maximal number of linearly independent 3-vectors of the set $\{(1, 0, -1), (0, -1, 2), (1, -1, 1)\}$.

Maximal number of linearly independent vectors

For any set of n-vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, the maximum number of linearly independent vectors is smaller or equal to n.

The number n it is called the dimension of the set of all n-vectors, usually denoted by $\mathbb{R}^n$, and any set of n linearly independent n-vectors is called a basis of $\mathbb{R}^n$.

A simple example of n-vectors is that of the coordinate n-vectors:

$$
\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, 0, \ldots, 0, 0) \\
\mathbf{e}_2 &= (0, 1, 0, 0, \ldots, 0, 0) \\
\vdots \quad & \quad \vdots \\
\mathbf{e}_n &= (0, 0, 0, 0, \ldots, 0, 1)
\end{aligned}
$$

The set of coordinate n-vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$ is linearly independent, and therefore is a basis of $\mathbb{R}^n$ known as canonical basis.

Matrix Rank

Let $\mathbf{A}$ a matrix of order $n \times m$. We can think of the column of $\mathbf{A}$ as n-vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m$, and the rows as m-vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$.

We call rank of $\mathbf{A}$, and denote it by $\text{rank}(\mathbf{A})$, to the maximum number of independent columns. The rank turns also to be the same as the maximum number of independient rows.

Properties of the rank

If $\mathbf{A}$ is $n \times m$ matrix, then:

- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}')$
- $\text{rank}(\mathbf{A}) \leq \min(n, m)$
- $\text{rank}(\mathbf{0}) = 0$
- $\text{rank}(\mathbf{I}_n) = n$
- $\text{rank}\left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) = \text{rank}(\mathbf{A})$

Elementary operations on a matrix

We call elementary operations on a matrix to the following types of operations:

- Interchanging two rows or columns.
- Multiply one row or column by a non-zero number.
- Add to a row or column another one multiplied by a number.

The rank of a matrix des not change under elementary operations, thus we can use them to transform a matrix into a simpler one to simplify the computation of the rank.

In particular, we can transform the matrix **A** into a matrix in reduced form:

$$\mathbf{A} \xrightarrow{\text{elementary operations}} \left(\begin{array}{c|c} \mathbf{I}_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \Rightarrow \operatorname{rank}(\mathbf{A}) = r$$

Elementary operations on a matrix

In general, it is not necessary to transform a matrix into its reduced form to obtain its range.

It is enough to realize elementary row operations to take it to triangular form.

Example:

The following matrix can te taken to triangular form by elementary row operations:

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 2 & -1 & 0 \\ 3 & -4 & 1 & -1 \\ 0 & 1 & -2 & -1 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It can be proven that a square matrix $\mathbf{A}$ of order n is invertible if and only if $\text{rank}(\mathbf{A}) = n$.

We can use elementary row operations to take $\mathbf{A}$ to the identity $\mathbf{I}_n$. If this same operations are applied to the identity $\mathbf{I}_n$, this is transformed into the inverse matrix:

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{I}_n \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{B} \end{array} \right)$$

Then $\mathbf{A}^{-1} = \mathbf{B}$.

Rank and Determinant

A square matrix $\mathbf{A}$ of order n has non-zero determinant, $|\mathbf{A}| \neq 0$, if and only if $\text{rank}(\mathbf{A}) = n$.

We can use elementary operations on $\mathbf{A}$ to transform it into a simpler matrix in order to compute the determinant, but we have to take into account the following:

- If two rows or columns are interchanged, the determinant changes its sign.
- If a row or column is multiplied by a non-zero number, the determinant is also multiplied by the same number.

Chapter 4: Matrices and Systems of Equations

Systems of Linear Equations

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \mathbf{A}^* = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{pmatrix}$$

Theorem

Let us consider the system of m linear equations with n variables described in matrix form as $Ax = b$, where A is a $m \times n$ matrix and b is a column vector with m components. Then, the system has a solution if and only if $rank(A) = rank(A^*)$. In addition, the solution is unique if and only if $rank(A) = n$.

Chapter 5: Diagonalization

Chapter 5: Diagonalization

Introduction

Diagonalization of matrices is a fundamental mathematical technique that plays a crucial role in various areas of mathematics, physics and engineering. At its core, diagonalization is a method used to transform a square matrix into a special form called a diagonal matrix. This process simplifies matrix operations and reveals important properties of the original matrix.

Chapter 5: Diagonalization

Diagonalization

Diagonalizable matrix

A square matrix $A \in \mathcal{M}_n$ is diagonalizable if there exist a matrix $C \in \mathcal{M}_n$ such that $Det(C) \neq 0$ and a diagonal matrix $D \in \mathcal{M}_n$ such that

$$A = CDC^{-1}$$

Properties

Let $A \in \mathcal{M}_n$ be a diagonalizable matrix such that $A = CDC^{-1}$ with $D \in \mathcal{M}_n$ diagonal and $C \in \mathcal{M}_n$ regular. Then:

- $\det(A) = \det(D)$
- $A^n = CD^nC^{-1}$

Chapter 5: Diagonalization

Eigenvalues and Eigenvectors

Let $A \in \mathcal{M}_n$ be a diagonalizable matrix such that $A = CDC^{-1}$ with $D \in \mathcal{M}_n$ diagonal and $C \in \mathcal{M}_n$ regular.

In such case, $C = (v_1|v_2|\cdots|v_n)$, where $v_1, v_2, \cdots, v_n$ are linearly independent vectors, and D can be writen as

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Since $AC = CD$, we have

$$AC = (Av_1|Av_2|\cdots|Av_n)$$
$$CD = (\lambda_1 v_1|\lambda_2 v_2|\cdots|\lambda_n v_n)$$

and $Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2,\ \cdots,\ Av_n = \lambda_n v_n$

Eigenvalues and Eigenvectors

Let $A \in \mathcal{M}_n$ be a square matrix. Given $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$, we say that λ is an eigenvalue for the eigenvector v if

$$Av = \lambda v$$

Note that if λ is an eigenvalue for the eigenvector v, $(A - \lambda I)v = 0$. And therefore, $Det(A - \lambda I) = 0$.

Characteristic Polynomial

Let $A \in \mathcal{M}_n$ be a square matrix. The Characteristic Polynomial is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Note that $\lambda \in \mathbb{R}$ is an eigenvalue if and only if $p(\lambda) = 0$

Multiplicity

Let $A \in \mathcal{M}_n$ be a square matrix and consider an eigenvalue $\lambda \in \mathbb{R}$.

- Algebraic Multiplicity. Number of times for which λ is a root of the characteristic polynomial.
- Geometric Multiplicity. $n - \text{rank}(A - \lambda I)$

Chapter 5: Diagonalization

Sufficient conditions to diagonalize

Sufficient conditions

Let $A \in \mathcal{M}_n$.

Properties

- If A is a symmetric matrix, i.e., $A = A^t$, A is diagonalizable,
- If A has n different Eigenvalues, A is diagonalizable,
- If A has n linearly independent Eigenvectors, A is diagonalizable,

Chapter 5: Diagonalization

Diagonalization and powers of a matrix

Property

Given $A \in \mathcal{M}_n$, if λ is an eigenvalue for the eigenvector v, then $A^k v = \lambda^k v$

Property

Let $A \in \mathcal{M}_n$ be a diagonalizable matrix and consider $v \in \mathbb{R}$. Let us assume that $\lambda_1, ..., \lambda_n$ are the eigenvalues and $v_1, ..., v_n$ are the eigenvectors. Then

- There exist $\alpha_1, ..., \alpha_n$ such that $v = \sum_{i=1}^n \alpha_i v_i$.
- $A^k v = \sum_{i=1}^n \alpha_i \lambda_i^k v_i$
- Consider a dominant eigenvalue $|\lambda_*| = \max_{i=1,...,n} |\lambda_i|$. Then, $A^k v = \lambda_*^k \sum_{i=1}^n \alpha_i \left(\frac{\lambda_i}{\lambda_*}\right)^k v_i$.
- If there is only one dominant eigenvalue, then

$$\lim_{k\to\infty} A^k v = \lim_{k\to\infty} \lambda_*^k \alpha_* v_*$$

Chapter 5: Diagonalization

Quadratic Form Classification

Quadratic Form

A Quadratic Form is a mapping $Q_A : \mathbb{R}^n \to \mathbb{R}$ defined as

$$Q_A(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where $A \in \mathcal{M}_n$ is a symmetric matrix.

Quadratic Forms Classification

Classification

Consider a quadratic Form $Q_A : \mathbb{R}^n \to \mathbb{R}$

- Positive Definite. $Q_A(x) > 0 \; \forall \; x \neq 0$. Equivalently, all the eigenvalues of A are positive.
- Negative Definite. $Q_A(x) < 0 \; \forall \; x \neq 0$. Equivalently, all the eigenvalues of A are negative.
- Positive Semi-definite. $Q_A(x) \geq 0 \; \forall \; x \neq 0$. Equivalently, all the eigenvalues of A are non-negative.
- Negative Semi-definite. $Q_A(x) \leq 0 \; \forall \; x \neq 0$. Equivalently, all the eigenvalues of A are non-positive.
- Indefinite. $Q_A(x_1) > 0$ and $Q_A(x_2) < 0$ for some x_1, x_2. Some eigenvalues of A are positive and others are negative.